where K is the smallest positive value of x in the equation

$$s_{a}^{x}=1.$$

Since $(t^{1}_{1}\, t^{1}_{2} \ldots\ldots t^{n}_{n} s_{\beta})^{K}=t^{1}_{1}\, t^{1}_{2} \ldots\ldots t^{1}_{n}\, t^{2}_{2}\, t^{2}_{3} \ldots\ldots t^{2}_{1} \ldots\ldots t^{n}_{n} t^{n}_{1} \ldots\ldots t^{n}_{n-1}$

we know that $t^{1}_{1}\, t^{2}_{2} \ldots\ldots t^{n}_{n} s_{\beta}$ may be multiplied by some substitution of G_1 so as to give for the new t's

$$t^{1}_{1}\, t^{2}_{2} \ldots\ldots t^{n}_{n}=1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Consider now the equations

$$(K^{1}_{1}\, K^{2}_{2} \ldots\ldots K^{n}_{n})^{-1} s_{\beta}\, K^{1}_{1}\, K^{2}_{2} \ldots\ldots K^{n}_{n}{}^{*}=$$

$$\overset{-1}{K^{n}_{n}} K^{n}_{1}\, \overset{-1}{K^{n-1}_{n-1}} K^{n-1}_{n} \ldots\ldots \overset{-1}{K^{1}_{1}}\, K^{1}_{2}\, s_{\beta}=t^{1}_{1}\, t^{2}_{2} \ldots\ldots t^{n}_{n} s_{\beta}\,.$$

We see directly that the following is a solution of the last equation if (A) is satisfied:

$$K^{1}_{1}=1,\ K^{2}_{2}=t^{2}_{1},\ K^{3}_{3}=t^{3}_{1}\, t^{3}_{2}, \ldots\ldots K^{n}_{n}=t^{n}_{1}\, t^{n}_{2} \ldots\ldots t^{n}_{n-1}.$$

Hence all the possible groups are conjugate to the one already given and our theorem is proved. This theorem may be employed with respect to the first subgroups as well as with respect to the entire groups.

In our next paper we shall consider the construction of the third and last class of groups, viz: the *primitive* groups.

[To be Continued.]

ON AN INTERESTING SYSTEM OF QUADRATIC EQUATIONS.

By DR. E. H. MOORE, University of Chicago, and EMMA C. ACKERMANN, Michigan State Normal School.

In C. Smith's Algebra, fourth edition, p. 134, are given for solution, examples 61, 62, 63, which are as follows (the third with a slight modification):

61. The roots of the equation $x^2+mx+m^2+a=0$ are x_1, x_2; show that $x_1{}^2+x_1x_2+x_2{}^2+a=0$.

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*This equation follows from the simpler one

$$(ts)^{-1}=s^{-1}t^{-1}$$

and this is true because if we multiply both members by ts we obtain an identity.

62. The roots of the equation $(x^2+1)(a^2+1)-max(ax-1)=0$ are x_1,x_2; show that $(x_1{}^2+1)(x_2{}^2+1)-mx_1x_2(x_1x_2-1)=0$.

63. The roots of the equation $a(x^2+mx+m^2)+bm^2x^2=0$ are x_1, x_2; show that $a(x_1{}^2+x_1x_2+x_2{}^2)+bx_1{}^2x_2{}^2=0$.

The equations possess the following properties: (1), the equation is of the second degree in the variable x and the constant a; (2), the roots x_1, x_2 of the equation are related to each other exactly as are the variable x and constant a.

We seek to generalize these theorems and formulate this problem:

To determine all quadratic equations of the form

$$f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})=0,$$

where the function $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})$ *is a symmetric function* $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})\equiv f(\overset{2}{\breve{m}}, \overset{2}{\breve{x}})$ *of its two arguments* x *and* m *of the second degree in each of them, characterized by the property that between the two roots* x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv 0$$

holds as an identity in m.

I. Let $f(\overset{2}{\breve{x}}, \overset{2}{\breve{m}})\equiv a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0.$

II. $\because f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv 0$, and x_1 and x_2 take the places of x and m,

$f(\overset{2}{\breve{x}}_1, \overset{2}{\breve{x}}_2)\equiv a+h(x_1+x_2)+bx_1x_2+g(x_1{}^2+x_2{}^2)+f(x_1{}^2x_2+x_2{}^2x_1)+cx_1{}^2x_2{}^2\equiv 0.$

We are to investigate now the conditions on the parameters a, b, c, f, g, h that must hold in order that $f(x_1,x_2)$ may as a function of m be identically 0. The problem then is not necessarily to prove $f(x_1,x_2)\equiv 0$ for all equations, but to find all equations for which it is true that $f(x_1,x_2)\equiv 0$.

III. Let $Kx^2+Lx+M=0$ be the original equation; x_1 and x_2 the roots; then $-K(x_1+x_2)=L$; $K(x_1x_2)=M$.

Comparing this equation with I:

$$K\equiv g+fm+cm^2.$$

$$L\equiv h+bm+fm^2.$$

$$M\equiv a+hm+gm^2.$$

IV. Transform equation in I to this form:

$$a+h(x+m)+(b-2g)xm+g(x+m)^2+f(xm)(x+m)+c(xm)^2=0.$$

V. Also equation in II to this form:

$$a+h(x_1+x_2)+(b-2g)x_1x_2+g(x_1+x_2)^2+f(x_1x_2)(x_1+x_2)+c(x_1x_2)^2\equiv 0.$$

VI. Multiply V by K^2:

$$aK^2+hK^2(x_1+x_2)+(b-2g)K^2x_1x_2+gK^2(x_1+x_2)^2 \\ +fK^2(x_1x_2)(x_1+x_2)+cK^2(x_1x_2)^2=0.$$

VII. VI becomes, by substituting for x_1x_2 and x_1+x_2 their values as given in III:

$$aK^2-hKL+(b-2g)KM+gL^2-fLM+cM^2\equiv 0$$

where K, L, M are given in terms of m in III.

Since VII is an identity in m, the coefficients of the different powers of m are each zero ; $\therefore$ the condition in VII requires that five polynomials homogeneous in a, b, c, f, g, h of degree three shall be zero. Since there are six letters, there are five ratios ; $\therefore$ there are five unknowns in five equations. This system of five cubic equations turns out to be extremely simple.

For, in VII, substituting for K, L, M their values involving m as given in III, collecting terms with reference to m, and using detached coefficients, we have :

1	m	m^2	m_3	m^4	
ag^2	$2afg$	af^2+2acg	$2acf$	ac^2	$\equiv aK^2$
$-gh^2$	$-(fh^2+bgh)$	$-(ch^2+bfh+fgh)$	$-(bch+f^2h)$	$-cfh$	$\equiv -hKL.$
$ag(b-2g)$	$(af+gh)(b-2g)$	$(ac+fh+g^2)(b-2g)$	$(ch+fg)(b-2g)$	$cg(b-2g)$	$\equiv(b-2g)KM$
gh^2	$2bgh$	$b^2g+2fgh$	$2bfg$	f^2g	$\equiv gL^2$
$-afh$	$-(abf+fh^2)$	$-(af^2+bfh+fgh)$	$-(f^2h+bfg)$	$-f^2g$	$\equiv -fLM.$
a^2c	$2ach$	ch^2+2acg	$2cgh$	cg^2	$\equiv cM^2.$

Simplifying and letting c_0, c_1 ...... be coefficient of m^0, m^1 ......

$$c_0\equiv a\{(b-g)g+ac-fh\}=0.$$

$$c_1\equiv 2h\{(b-g)g+ac-fh\}=0.$$

$$c_2\equiv(b+2g)\{(b-g)g+ac-fh\}=0.$$

$$c_3\equiv 2f\{(b-g)g+ac-fh\}=0.$$

$$c_4\equiv c\{(b-g)g+ac-fh\}=0.$$

This means that given $f(x,m)=0$ as in I, then $f(x_1, x_2)\equiv 0$, if, and only if, either $a=2h=b+2g=2f=c=0$, or $(b-g)g+ac-fh=0$; the second alternative is *one* condition, homogeneous, of degree two, between the six homogeneous parameters. Therefore,

All quadratic equations of the form

$$a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$$

(in which the first member is a symmetric function $f(\underset{\smile}{\overset{2}{x}}, \underset{\smile}{\overset{2}{m}})\equiv f(\underset{\smile}{\overset{2}{m}}, \underset{\smile}{\overset{2}{x}})$ *of its two arguments* x *and* m *of the second degree in each of them), whose parameters are related by the equation*

$$(b-g)g+ac-fh=0,$$

—and, apart from the relatively trivial equation

$$g(x^2-2mx+m^2)=0,$$

only those equations whose parameters are so related—are characterized by the property that between the two roots x_1, x_2 *which are functions of* m *the relation*

$$f(\underset{\smile}{\overset{2}{x_1}}, \underset{\smile}{\overset{2}{x_2}})\equiv 0$$

holds as an identity in m.

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QUADRATURE OF THE CIRCLE.

By WILLIAM E. HEAL, A. M., Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

The problem of the quadrature of the circle, or what amounts to the same thing, drawing a straight line equal in length to the circumference of a given circle, occupied the attention of mathematicians at a very early date. Long before the time of Archimedes, geometers had attacked the problem with but one result: failure. And for more than twenty centuries mathematicians have been struggling with the problem. Many claimed to have solved it, but their analysis has been, in every case, found to be fatally defective. After centuries of attempt and failure mathematicians began to suspect that the problem might not admit of solution. James Gregory was the first to attempt a proof of the impossibility of the quadrature of the circle. In the opinion of Montucla he succeeded; but later mathematicians have not so decided. Not a score of years have passed since a rigid proof was given that the solution of the problem is really impossible under the conditions usually understood: that is, by the use of the rule and compass only.